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AGE DEPENDENT MINIMAL REPAIR

by

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AGE DEPENDENT IMPERFECT REPAIR

by

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ABSTRACT

A stochastic model is developed to describe the operation in time of the following maintained system setting. A piece of equipment is put in operation at time 0. Each time it fails, a maintenance action is taken which, with probability $p(t)$, is a complete repair or, with probability $q(t) = 1 - p(t)$, is a minimal repair, where t is the age of the equipment in use at the failure time. It is assumed that complete repair restores the equipment to its good as new condition, that minimal repair restores the equipment to its condition just prior to failure and that both maintenance actions take negligible time.

If the equipment's life distribution F is a continuous function, the successive complete repair times are shown to be a renewal process with interarrival distribution $F_p(t) = 1 - \exp\{-\int_0^t p(x)\bar{F}^{-1}(x)\ F(dx)\}$ for $t \geq 0$. Preservation and monotone properties of the model extending the results of Brown and Proschan (1980) are obtained.

AMS 1970 subject classification: Primary 62N05; Secondary 60K10.

Key Words: Minimal repair, life distributions, renewal processes.

1. INTRODUCTION

Most of the literature concerning the stochastic behavior of repairable systems assumes that repair of a unit provides a functioning one which is as good as new . However, the technical convenience of this assumption and its implications were criticized by many authors on the grounds that repair in many practical instances only restores a unit to its functioning condition just prior to failure. See for example Ascher and Feingold (1979) and references therein.

This discussion stimulated renewed interest in modeling repairable systems taking into account such "minimal repair actions". New research efforts in this particular branch of reliability theory include the works of Blumenthal et al (1976), Brown and Proschan (1980) and Balaban and Singpurwalla (1981) among others.

The failure process studied in this paper models the following maintained system setting. A piece of equipment is put in operation at time $t=0$. Each time it fails, a maintenance action is taken which, with probability $p(t)$, is a complete repair or, with probability $q(t) = 1 - p(t)$, is a minimal repair, where t is the age at failure of the equipment under maintenance.

Since availability results are not pursued in this paper, only operating time will be recorded. This is equivalent to assuming that maintenance is executed in negligible time. It is also assumed that complete repairs restore failed items to their good as new condition in such a way that the times between successive complete repairs are independent and identically distributed.

The formal development of the model is given in an appendix where the basic facts are established. In Section 2 we show that some aging properties of the equipment's life distribution are inherited by the distribution of the time between successive complete repairs under suitable monotonicity of the function p. A counterexample is also given to the conjecture of Brown and Proschan (1980) that the NBUE aging property is also inherited when the function p is constant.

In Section 3 some inequalities and further properties of the model are developed which, as in Section 2, extend results obtained by Brown and Proschan (1980).

2. PRESERVATION OF AGING PATTERNS

To develop our first results, we start with a brief informal description of the model; a rigorous development is given in the appendix. We consider the maintained system setting in which a piece of equipment is put in operation at time zero and every time a failure occurs it is repaired. If t is the equipment's age at failure, with probability $p(t)$ it is restored to its good as new condition (complete repair) and with probability $1-p(t)$ it is restored to its condition just prior to failure (minimal repair). We will assume throughout that the good as new condition of our equipment is described by a survival distribution $\bar{F}$ which is a continuous function and such that $\bar{F}(t) > 0$ for all $t \geq 0$. We also recall the following two facts:

(i) if a piece of equipment with survival distribution $\bar{F}(x)$ fails at age $t \geq 0$ and undergoes minimal repair, the functioning equipment obtained has survival distribution $\bar{F}(x|t) = \bar{F}^{-1}(t)\ \bar{F}(x+t)$;

(ii) repairs take negligible time.

If only minimal repair actions are taken, the failure process is well known to be a point process $\{S_n;\ n \geq 1\}$ on $\mathbb{R}_+$, whose corresponding counting process $\{N(t);\ t \geq 0\}$ is a nonhomogeneous Poisson process with mean function $M(t) = EN(t) = -\log \bar{F}(t)$ for $t \geq 0$. Letting $p: \mathbb{R}_+ \to [0,1]$ be measurable and setting $q = 1-p$, we now consider a sequence $\{Z_n;\ n \geq 1\}$ of (conditionally) independent Bernoulli trials such that $P\{Z_n = 1\} = p(S_{n-1})$, for $n \geq 1$, with $S_0 \equiv 0$. As long as failures are observed, minimal repairs are successively performed; however, at the time the first success is observed, a complete repair is executed and we begin afresh since we have the equipment restored to its good as new condition. This means that our general failure process regenerates every time a complete repair is made, so that successive complete repair times form a (possibly delayed) renewal process. Assuming that we start out with new equipment, we have indeed a standard renewal process and in this section we will examine preservation properties of its interarrival distribution.

Let Y denote the time until the first perfect repair starting with a new item. The relevant results about Y, as derived in the appendix, are restated for convenience in the following theorem.

(2.1) Theorem. Y is finite with probability one if and only if

$$\int_0^\infty p(y)\bar{F}^{-1}(y)F(dy) = +\infty .$$

In this case we have

$$(2.2) \qquad P\{Y > t\} = E\Big[\prod_{i=1}^{N(t)} q(S_i)\Big] = \exp\Big\{-\int_0^t p(y)\bar{F}^{-1}(y)F(dy)\Big\} \text{ for } t \geq 0,$$

with the convention that $\prod_{i=1}^{N(t)} q(S_i) \equiv 1$ on $\{N(t) = 0\}$. Also

(2.3) $$E\,Y = g(0) + \sum_{n=1}^{\infty} E[g(S_n) \prod_{i=1}^{n} q(S_i)]\,,$$

where $g(t) = \int_0^{\infty} \bar{F}(x|t)\,dx$ for $t \geq 0$.

<u>Proof</u>. Since the first two assertions are proved in the appendix, we proceed to validate equality (2.3).

$$\begin{aligned} E\,Y &= \int_0^{\infty} P\{Y > t\}\,dt = \int_0^{\infty} E[\prod_{i=1}^{n} q(S_i);\ N(t) \geq 1]\,dt \\ &= \int_0^{\infty} P\{N(t) = 0\}\,dt + \sum_{n=1}^{\infty} \int_0^{\infty} E[\prod_{i=1}^{n} q(S_i);\ N(t) = n]\,dt \\ &= \int_0^{\infty} P\{S_1 > t\}\,dt + \sum_{n=1}^{\infty} \int_0^{\infty} E[\prod_{i=1}^{n} q(S_i);\ S_n \leq t < S_{n+1}]\,dt \\ &= \int_0^{\infty} \bar{F}(t)\,dt + \sum_{n=1}^{\infty} E[\prod_{i=1}^{n} q(S_i) \int_0^{\infty} I_{[S_n, S_{n+1})}(t)\,dt] \\ &= g(0) + \sum_{n=1}^{\infty} E[(S_{n+1} - S_n) \prod_{i=1}^{n} q(S_i)]\,. \end{aligned}$$

But

$$E[(S_{n+1} - S_n) \prod_{i=1}^{n} q(S_i)] = E[\prod_{i=1}^{n} q(S_i)\ E[S_{n+1} - S_n | S_1, \ldots, S_n]],$$

and since

$$E[S_{n+1} - S_n | S_1, \ldots, S_n] = \int_0^{\infty} \bar{F}(x|S_n)\,dx = g(S_n)\,,$$

we obtain

$$E\,Y = g(0) + \sum_{n=1}^{\infty} E[g(S_n) \prod_{i=1}^{n} q(S_i)]\,.$$

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(2.4) Remarks.

(i) We shall henceforth assume that $\int_0^\infty p(y)\bar{F}^{-1}(y)F(dy) = +\infty$.

(ii) For convenience we shall sometimes denote the distribution function of Y by F_p.

(iii) If F has a failure rate function r, then F_p has failure rate function given by $r_p(t) = p(t)\ r(t)$ for $t \geq 0$.

(iv) We will use the convention that $\prod_{i=1}^{n} (\cdot) = 1$ for $n = 0$ throughout the paper.

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We now derive the preservation results.

(2.5) Theorem. If p is increasing (decreasing), then F_p is IFR, IFRA, NBU or DMRL (DFR, DFRA, NWU or IMRL) whenever F is.

Proof. Let $R_p(t) = \int_0^t p(y)\bar{F}^{-1}(y)F(dy)$ be the hazard function of F_p, and assume that p is increasing. To show that F_p is IFR whenever F is, we need only show that R_p is convex. Now if $p(t) = I_{(a,\infty)}(t)$ or $p(t) = I_{[a,\infty)}(t)$ and R is the hazard function of F, then

$$R_p(t) = \begin{cases} 0 & \text{if } 0 \leq t < a\,, \\ R(t) - R(a) & \text{if } a \leq t\,, \end{cases}$$

which is easily seen to be convex in $t \geq 0$. Since any nonnegative increasing function p can be obtained as the pointwise limit of nonnegative linear combinations of such indicator functions, it follows that R_p is convex in $\mathbb{R}_+$ for any increasing nonnegative function p.

The same technique is used to prove the IFRA and NBU cases when p is increasing, by verifying star-shapedness and super-additivity of R_p, respectively.

We now assume that F is DMRL. By definition, $g(t) = \int_0^\infty \bar{F}(x|t)\,dx$ is decreasing in $t \geq 0$, and to show that F_p is *DMRL we need only show that* $g_p(t) = \int_0^\infty \bar{F}_p(x|t)\,dx$ is also decreasing in $t \geq 0$. If we observe that

$$\bar{F}_p(x|t) = \exp\{- \int_t^{t+x} p(y)\bar{F}^{-1}(y)F(dy)\}$$

$$= \exp\{- \int_0^x p(y+t)\bar{F}^{-1}(y|t)F(dy|t) \quad \text{for } x \geq 0 ,$$

we see that the conditional survival distribution $\bar{F}_p(x|t)$ is the same as the survival distribution of a random variable Y' obtained in the same fashion as Y but replacing $p(y)$ by $p'(y) = p(y+t)$ and $F(y)$ by $F'(y) = F(y|t)$. Thus, if we denote the corresponding point process of minimal repair times by $\{S_n'; n \geq 1\}$ and the corresponding counting process by $\{N'(u); u \geq 0\}$, we deduce from (2.3) that

$$g_p(t) = EY' = g'(0) + \sum_{n=1}^\infty E[g'(S_n') \prod_{i=1}^n q'(S_i')] \quad \text{for } t \geq 0,$$

where

$$g'(s) = \int_0^\infty \bar{F}'(x|s)\,dx = \int_0^\infty \bar{F}(x|s+t)\,dx = g(s+t) \qquad \text{for } s \geq 0.$$

So,

$$g_p(t) = g(t) + \sum_{n=1}^\infty E[g(S_n'+t) \prod_{i=1}^n q(S_i'+t)] \qquad \text{for } t \geq 0.$$

Since it can be shown that the joint distribution of $(S_1',\ldots,S_n')$ is the same as that of $(S_{N(t)+1}-t,\ldots,S_{N(t)+n}-t)$ (see (2.6)), it follows that

$$g_p(t) = g(t) + \sum_{n=1}^\infty E[g(S_{N(t)+n}) \prod_{i=1}^n q(S_{N(t)+i})] \quad \text{for } t \geq 0.$$

Because $N(t)$ increases deterministically with t and g and q are decreasing, it follows that $g_p(t)$ is decreasing in $t \geq 0$ and the proof is complete.

The dual results for decreasing p are handled similarly.

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(2.6) Remarks.

(i) Note that $S_{N(t)+j}$ is the time of the jth jump after t of the process $\{N(u);\ u \geq 0\}$.

(ii) The joint distribution of $(S_1', \ldots, S_n')$ is the same as the joint distribution of $(S_{N(t)+1}-t, \ldots, S_{N(t)+n}-t)$. It is sufficient to show this only in the case $n=2$. Letting $v \geq u \geq 0$, we have

$$
\begin{aligned}
P\{S_{N(t)+1} - t > u,\ S_{N(t)+2} - t > v\} \\
&= \sum_{j=0}^{\infty} P\{S_{j+1} > u+t,\ S_{j+2} > v+t,\ N(t) = j\} \\
&= \sum_{j=0}^{\infty} P\{N(u+t) \leq j,\ N(v+t) \leq j+1,\ N(t) = j\} \\
&= \sum_{j=0}^{\infty} P\{N(t) = j,\ N(t+u) - N(t) = 0,\ N(t+v) - N(t+u) \leq 1\} \\
&= \quad P\{N(t+u) - N(t) = 0,\ N(t+v) - N(t+u) \leq 1\} \sum_{j=0}^{\infty} P\{N(t) = j\} \\
&= \quad P\{N'(u) = 0,\ N'(v) - N'(u) \leq 1\} \\
&= \quad P\{S_1' > u,\ S_2' > v\}\ .
\end{aligned}
$$

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We lastly consider the NBUE case when p is increasing. It was conjectured in Brown and Proschan (1980) that the preservation result would also be valid for their model (p constant). We show that this is false through the following counterexample.

(2.7) Counterexample:

Consider the following continuous survival distribution

$$\bar{F}(x) = \begin{cases} 1 & 0 \leq x \leq 15 \\ \text{linear} & 15 \leq x \leq z \\ 2\mathrm{x}10^{-4} & z \leq x \leq 25 \\ \text{linear} & 25 \leq x \leq 35 \\ 0 & 35 \leq x \end{cases}$$

where z is chosen such that $(0.5)\ (1 - 2\mathrm{x}\,10^{-4})\ (z - 15) = 5\mathrm{x}\ 10^{-7}$.

Simple calculations show that this distribution is NBUE but not NBU. However, for p = 0.5, F_p is not NBUE.

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3. INEQUALITIES

Let us define

$$\mu(p) = E\,Y = \int_0^\infty \bar{F}_p(x)\,dx$$

and recall from (2.3) that

(3.1) $$\mu(p) = g(0) + \sum_{n=1}^{\infty} E[g(S_n) \prod_{i=1}^{n} q(S_i)]\ .$$

(3.2) Theorem.

(i) Let F be NBUE. Then

$$\mu(p) \leq \mu(1) \int_0^\infty \exp\{- \int_0^x p(y)\bar{F}^{-1}(y)F(dy)\}\ \bar{F}^{-1}(x)F(dx)\ .$$

(ii) Let $p_1(t) \leq p_2(t)$ for all $t \geq 0$, and assume that F_{p_2} is NBUE. Then

$$\mu(p_1) \leq \mu(p_2) \int_0^\infty p_2(x)\ \exp\{- \int_0^x p_1(y)\bar{F}^{-1}(y)F(dy)\}\bar{F}^{-1}(x)F(dx).$$

In particular, if p_1 and p_2 are proportional, then

$$p_1(t)\ \mu(p_1) \leq p_2(t)\ \mu(p_2) \quad \text{for all } t \geq 0.$$

<u>Proof</u>.

(i) Since $g(t) \leq g(0) = \mu(1)$ it follows from (3.1) that

$$\mu(p) \leq \mu(1)\ [1+\sum_{n=1}^{\infty} E[\prod_{i=1}^{n} q(S_i)]]. \tag{3.3}$$

Now as developed in the appendix,

$$E[\prod_{i=1}^{n} q(S_i)] = \int_0^{\infty} q(x_1)\bar{F}^{-1}(x_1) \int_{x_1}^{\infty} q(x_2)\bar{F}^{-1}(x_2)\ldots\int_{x_{n-1}}^{\infty} q(x_n)F(dx_n)\ldots F(dx_1)$$
$$= \int_0^{\infty} q(x)\ \frac{1}{(n-1)!}\ \{\int_0^x q(y)\bar{F}^{-1}(y)F(dy)\}^{n-1}F(dx)$$

for $n \geq 1$, and hence

$$\sum_{n=1}^{\infty} E[\prod_{i=1}^{n} q(S_i)] = \int_0^{\infty} q(x)\ \exp\{\int_0^x q(y)\bar{F}^{-1}(y)F(dy)\}F(dx) \tag{3.4}$$
$$= \int_0^{\infty} \exp\{-\int_0^x p(y)\bar{F}^{-1}(y)F(dy)\}\bar{F}^{-1}(x)F(dx) - 1$$

The result now follows from (3.3) and (3.4).

(ii) It will now be convenient to denote the expected value of Y by $\mu(p;F)$ instead of $\mu(p)$ to stress also its dependence on the distribution F. Since

$$\int_0^x p(y)\bar{F}^{-1}(y)F(dy) = \int_0^x \bar{F}_p^{-1}(y)F_p(dy) \quad \text{for } x \geq 0,$$

we can write $\mu(p;F) = \mu(1;F_p)$, and similarly one can show that if $p_1(t) \leq p_2(t)$ for all $t \geq 0$, then $\mu(p_1;F) = \mu(p_1/p_2;F_{p_2})$. Hence, if F_{p_2} is NBUE, it follows from (i) that

$$\mu(p_1) = \mu(p_1;F) = \mu(p_1/p_2;F_{p_2})$$
$$\leq \mu(1;F_{p_2})\int_0^{\infty} \exp\{-\int_0^x \frac{p_1(y)}{p_2(y)}\ \bar{F}_{p_2}^{-1}(y)F_{p_2}(dy)\}\bar{F}_{p_2}^{-1}(x)F_{p_2}(dx)$$
$$= \mu(p_2)\int_0^{\infty} p_2(x)\ \exp\{-\int_0^x p_1(y)\bar{F}^{-1}(y)F(dy)\}\bar{F}^{-1}(x)F(dx).$$

This proves (ii), and the last assertion is a direct consequence of the inequality.

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(3.5) Remark. If ν^* denotes the number of failures occurring up to and including time Y, we have $P\{\nu^*-1 \geq n\} = E[\prod_{i=1}^{n} q(S_i)]$, for $n \geq 1$, and in (3.4) we have shown that

$$E\,\nu^* = \int_0^\infty \exp\{-\int_0^x p(y)\bar{F}^{-1}(y)F(dy)\}\bar{F}^{-1}(x)F(dx).$$

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We now consider the entire failure process associated with p starting with new equipment having life distribution F. For each $t \geq 0$, define the following quantities: $A_p(t)$ is the age of the equipment in use at time t (i.e., the time from the last perfect repair until t); $Z_p(t)$ is the waiting time from t to the next perfect or imperfect repair; and $Z_p^*(t)$ is the waiting time from t to the next perfect repair. It is clear that

(3.6) $$P\{Z_p(t) > x\} = \int_0^\infty \bar{\bar{F}}(x|y)\ P\{A_p(t) \in dy\} \quad \text{for } x \geq 0,$$

and since successive complete repair times form a standard renewal process, we have the following theorem.

(3.7) Theorem. Let F be DFR and p be decreasing. Then $A_p(t)$, $Z_p(t)$ and $Z_p^*(t)$ are stochastically increasing in t. Furthermore the standard renewal process of complete repair times has a version of its renewal density, m_p^*, which is decreasing on $\mathbb{R}_+$.

Proof. Since F_p is also DFR, the results follow immediately from Theorem 3 of Brown (1980) and equality (3.6).

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We also recall from standard renewal theory that

(3.8) $$\lim_{t\to\infty} P\{A_p(t) \in dy\} = \mu^{-1}(p)\bar{F}_p(y)\,dy,$$

where the above limit is in distribution. The following result is readily verified.

(3.9) Theorem. Let F be IFR (DFR), and let Z_p have the asymptotic distribution of $Z_p(t)$ as $t\to\infty$. If $p_1(t) \le p_2(t)$ for all $t \ge 0$, we have

$$Z_{p_1} \overset{st}{\le} (\overset{st}{\ge})\, Z_{p_2}.$$

Furthermore, if F is absolutely continuous with failure rate function r and h_p denotes the failure rate function of Z_p, we have

$$h_{p_1}(t) \ge (\le)\, h_{p_2}(t) \quad \text{for all } t \ge 0.$$

Proof. First notice that $\bar{F}_{p_2}(x)/\bar{F}_{p_1}(x) = \exp\{-\int_0^x [p_2(y)-p_1(y)]\bar{F}^{-1}(y)F(dy)\}$ is monotone decreasing in x. It follows from this fact that if γ is a nonnegative function for which $\int_0^\infty \gamma(y)\bar{F}_{p_1}(y)dy < \infty$, then the densities $\gamma_{p_i}(y) = c_i\ \gamma(y)\bar{F}_{p_i}(y)$ for $y \ge 0$ and $i = 1,2$, satisfy the relation

(3.10) $$\int_t^\infty \gamma_{p_1}(y)\,dy \ge \int_t^\infty \gamma_{p_2}(y)\,dy \quad \text{for all } t \ge 0.$$

Taking $\gamma(y) = 1$ for all $y \ge 0$, it follows from (3.6), (3.8) and (3.10) that

$$\begin{aligned} P\{Z_{p_1} > x\} &= \int_0^\infty \bar{F}(x|y)\gamma_{p_1}(y)dy \\ &\le \int_0^\infty \bar{F}(x|y)\gamma_{p_2}(y)dy = P\{Z_{p_2} > x\} \text{ for } x \ge 0, \end{aligned}$$

since $\bar{F}(x|y)$ is decreasing in y. This proves the first assertion.

If F is also absolutely continuous with failure rate function r, and $\gamma(y;x) = \bar{F}(x|y)$, then the failure rate function of Z_{p_i}, $i = 1,2$, is given by

$$h_{p_i}(x) = \frac{\int_0^\infty r(x+y)\bar{F}(x|y)\mu^{-1}(p_i)\bar{F}_{p_i}(y)dy}{\int_0^\infty \bar{F}(x|y)\mu^{-1}(p_i)\bar{F}_{p_i}(y)dy}$$

$$= \int_0^\infty r(x+y)\gamma_{p_i}(y;x)dy \quad \text{for } x \geq 0.$$

Here $\gamma_{p_i}(y;x) = \bar{F}(x|y)\bar{F}_{p_i}(y)/\int_0^\infty \bar{F}(x|z)\bar{F}_{p_i}(z)dz$ and is obtained from $\gamma(y;x)$ in the same fashion as $\gamma_{p_i}(y)$ is obtained from $\gamma(y)$. The second assertion now follows from (3.10) and the monotonicity of r.

The dual results follow similarly.

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(3.11) Note. In Theorem 3.9 notice that

$$\frac{P\{Z_p > x+z\}}{P\{Z_p > x\}} = \frac{\int_0^\infty \bar{F}(x+z|y)\mu^{-1}(p)\bar{F}_p(y)dy}{\int_0^\infty \bar{F}(x|y)\mu^{-1}(p)\bar{F}_p(y)dy}$$

$$= \int_0^\infty \bar{F}(z|x+y)\gamma_p(y;x)dy$$

which decreases in x for each fixed z. Hence, Z_p is also IFR.

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From Theorem 3.9 it follows that

$$E\,Z_{p_1} \leq E\,Z_{p_2}$$

if $p_1(t) \leq p_2(t)$ for all $t \geq 0$ and F is IFR. This conclusion holds however under weaker conditions on F.

(3.12) Proposition. Let Z_p be as in Theorem 3.9. If $p_1(t) \leq p_2(t)$ for all $t \geq 0$ and F is DMRL (IMRL), then

$$E\,Z_{p_1} \leq (\geq)\, E\,Z_{p_2} .$$

Proof. If we let $\gamma_p(y) = \mu^{-1}(p)\bar{F}_p(y)$ for $y \geq 0$, then

$$\begin{aligned} E\,Z_p &= \int_0^\infty \int_0^\infty \bar{F}(x|y)\ \gamma_p(y)\,dy\,dx \\ &= \int_0^\infty \gamma_p(y)\ [\int_0^\infty \bar{F}(x|y)dx]dy \\ &= \int_0^\infty \gamma_p(y)g(y)\,dy , \end{aligned}$$

where g is as in Theorem 2.1. Since

$$\int_t^\infty \gamma_{p_1}(y)dy \geq \int_t^\infty \gamma_{p_2}(y)dy \quad \text{for all } t \geq 0$$

and g is decreasing, the result follows.

The dual result follows similarly.

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4. APPENDIX

In this appendix we formally construct the model for the maintained system setting described in Section 2. A similar model was discussed in Savits (1976) in the context of an age dependent branching process.

Since a complete repair at any failure point restores the equipment to its good as new condition and regenerates the failure process, it suffices to model the failure process up to its first regeneration point. This will be done through a sequence $\{(S_n, Z_n); n \geq 1\}$ of random variables, where $S_n \geq 0$ denotes the time of the n-th repair and $Z_n = 0$ or 1 indicates a minimal or a complete repair at time S_{n-1} ($S_0 \equiv 0$, $Z_1 = 0$), respectively. We only consider this process up to the random time $S_{\nu-1}$, where $\nu = \inf\{n \geq 1: Z_n = 1\}$. According to the Ionescu-Tulcea Theorem, c.f. Neveu (1965, p. 162), we need only specify the n-th step transition probability functions and the initial distribution.

Let F be a life distribution function such that $F(0) = 0$. In order to avoid technical difficulties we assume that F is a continuous function. Although it is not necessary, we shall also assume for convenience that $F(t) < 1$ for all $t \geq 0$. Let $p: \mathbb{R}_+ \to [0,1]$ be measurable and set $q(t) = 1 - p(t)$ for $t \geq 0$.

The nth-step transition probability function is given by

$$\text{(A.1)} \quad P\{S_{n+1} > t, Z_{n+1} = z \mid S_1, Z_1, \ldots, S_n, Z_n\} = p(S_n)^z q(S_n)^{1-z} \bar{F}(S_n \vee t)/\bar{F}(S_n)$$

for $n = 1, 2, \ldots$ and the initial distribution is

$$\text{(A.2)} \qquad P\{S_1 > t, Z_1 = z\} = \bar{F}(t)\, \delta_{\{0\}}(z).$$

From (A.1) and (A.2), it is easy to check by induction that

(A.3) $Ef(S_1,Z_1,\ldots,S_n,Z_n) =$

$$\sum_{z_1,\ldots,z_n=0}^{1} \int_0^{\infty} \bar{F}^{-1}(s_1) \int_{s_1}^{\infty} \bar{F}^{-1}(s_2) \ldots \int_{s_{n-2}}^{\infty} \bar{F}^{-1}(s_{n-1}) \times$$

$$\int_{s_{n-1}}^{\infty} \prod_{i=2}^{n} p(s_{i-1})^{z_i} q(s_{i-1})^{1-z_i} \; f(s_1,0,s_2,z_2,\ldots,s_n,z_n) F(ds_n) \ldots F(ds_1)$$

for any nonnegative measurable function $f(s_1,z_1,\ldots,s_n,z_n)$ and $n \geq 2$.

Now, let $\{N(t);\ t \geq 0\}$ be the counting process corresponding to $\{S_n:\ n \geq 1\}$, which is defined by $N(t) = \sum_{n=1}^{\infty} I_{[0,t]}(S_n)$.

(A.4) Theorem. $\{N(t),\ t \geq 0\}$ is a nonhomogeneous Poisson process with mean function $M(t) = E\,N(t) = -\log \bar{F}(t)$.

Proof. From Çinlar (1975, p. 96) it suffices to show that $M(S_1), M(S_2)-M(S_1), \ldots, M(S_{n+1})-M(S_n), \ldots$ is a sequence of independent identically exponentially distributed random variables with parameter $\lambda = 1$. If $n \geq 2, t_1 > 0, \ldots, t_{n+1} > 0$ and we let

$f(s_1,z_1,\ldots,s_n,z_n) =$

$$P\{M(S_1) > t_1, M(S_2)-M(S_1) > t_2, \ldots, M(S_{n+1})-M(S_n) > t_{n+1} \mid S_1=s_1, Z_1=z_1, \ldots, S_n=s_n, Z_n=z_n\},$$

straightforward calculations give

$f(s_1,z_1,\ldots,s_n,z_n) =$

$$P\{S_1 > G(e^{-t_1}), S_2 > G(e^{-t_2}\bar{F}(s_1)), \ldots, S_{n+1} > G(e^{-t_{n+1}}\bar{F}(s_n)) \mid S_1=s_1, Z_1=z_1, \ldots, S_n=s_n, Z_n=z_n\},$$

where G denotes the left continuous inverse of $\bar{F}$. From (A.1) we have

$$f(s_1,z_1,\ldots,s_n,z_n) = \begin{cases} 0 & \text{if } s_i \le G(e^{-t_i}\,\bar{F}(s_{i-1})) \text{ for some } i=1,\ldots,n\ (s_0=0), \\ \dfrac{\bar{F}(G(e^{-t_{n+1}}\,\bar{F}(s_n)\vee s_n))}{\bar{F}(s_n)} = e^{-t_{n+1}} & \text{otherwise}, \end{cases}$$

since $G(e^{-t_{n+1}}\,\bar{F}(s_n)) \ge G(\bar{F}(s_n)) \ge s_n$. Finally, from (A.3)

$$P\{M(S_1) > t_1,\ M(S_2)-M(S_1)>t_2,\ldots,M(S_{n+1})-M(S_n)>t_{n+1}\}=Ef(S_1,Z_1,\ldots,S_n,Z_n)$$

$$= e^{-t_{n+1}} \int_0^\infty \bar{F}^{-1}(s_1)I\{s_1>G(e^{-t_1})\} \int_{s_1}^\infty \bar{F}^{-1}(s_2)I\{s_2>G(e^{-t_2}\,\bar{F}(s_1))\}\ldots\times$$

$$\int_{s_{n-2}}^\infty \bar{F}^{-1}(s_{n-1})I\{s_{n-1}>G(e^{-t_{n-1}}\,\bar{F}(s_{n-2}))\}\int_{s_{n-1}}^\infty I\{s_n>G(e^{-t_n}\,\bar{F}(s_{n-1}))\}F(ds_n)\ldots F(ds_1)$$

$$= \exp\{-\sum_{i=1}^{n+1} t_i\},$$

as desired.

///

As observed, we are interested in the behavior of the sequence $\{(S_n,Z_n);\ n\ge 1\}$ only up to time $Y = S_{\nu-1}$, where $\nu = \inf\{n\ge 1;\ Z_n = 1\}$ with $\inf\phi = +\infty$. We note that $\nu \ge 2$ since $Z_1 = 0$; further results are contained in the following theorem.

(A.5) Theorem.

(a) $P\{\nu=n+1 | S_1,Z_1,\ldots,S_n,Z_n) = p(S_n) \prod_{i=1}^{n} I\{Z_i=0\}$.

(b) $P\{Y>t;\nu<+\infty\} = E[\prod_{i=1}^{N(t)} q(S_i)]$.

(c) $P\{\nu<+\infty\} = 1$ if and only if $\int_0^\infty p(y)\bar{F}^{-1}(y)F(dy) = +\infty$.

(d) If $P\{\nu<+\infty\} = 1$, then

$$P\{Y>t\} = \exp\{-\int_0^t p(y)\bar{F}^{-1}(y)F(dy)\}.$$

Proof.

(a) $$P\{\nu=n+1|S_1,Z_1,\ldots,S_n,Z_n\}=P\{Z_1=\ldots=Z_n=0,Z_{n+1}=1|S_1,Z_1,\ldots,S_n,Z_n\}$$
$$=\prod_{i=1}^{n} I\{Z_i=0\}P\{Z_{n+1}=1|S_1,Z_1,\ldots,S_n,Z_n\}$$
$$=p(S_n)\prod_{i=1}^{n} I\{Z_i=0\}\,.$$

(b) $$P\{Y>t;\ \nu<+\infty\}=\sum_{n=1}^{\infty} P\{Y>t,\ \nu=n+1\}$$
$$=\sum_{n=1}^{\infty} P\{S_n>t,\ \nu=n+1\}$$
$$=\sum_{n=1}^{\infty} E[P\{\nu=n+1|S_1,Z_1,\ldots,S_n,Z_n\};S_n>t]$$
$$=\sum_{n=1}^{\infty} E[p(S_n)\prod_{i=1}^{n} I\{Z_i=0\};\ S_n>t]$$
$$=\sum_{n=1}^{\infty} E[p(S_n)\prod_{i=2}^{n} q(S_{i-1});\ S_n>t]\,.$$

On the other hand,

$$E[\prod_{i=1}^{N(t)} q(S_i)]=\sum_{n=0}^{\infty} E[\prod_{i=1}^{n} q(S_i);\ N(t)=n]$$
$$=\sum_{n=0}^{\infty} E[\prod_{i=1}^{n} q(S_i);\ S_n\le t<S_{n+1}]$$
$$=\sum_{n=0}^{\infty} E[\prod_{i=1}^{n} q(S_i);\ S_{n+1}>t]-\sum_{n=0}^{\infty} E[\prod_{i=1}^{n} q(S_i);S_n>t]$$
$$=\sum_{n=0}^{\infty} E[p(S_{n+1})\prod_{i=1}^{n} q(S_i);\ S_{n+1}>t]$$
$$=\sum_{n=1}^{\infty} E[p(S_n)\prod_{i=2}^{n} q(S_{i-1});\ S_n>t]\,.$$

(c) and (d)

$$P\{Y>t,\ \nu=n+1\} = P\{S_n>t,\ Z_1=Z_2=\ldots=Z_n=0,\ Z_{n+1}=1\}$$

$$=\int_0^\infty \bar{F}^{-1}(s_1)\int_{s_1}^\infty \ldots \int_{s_{n-1}}^\infty \bar{F}^{-1}(s_n)\int_{s_n}^\infty p(s_n)\prod_{i=1}^{n-1} q(s_i)I\{s_n>t\}F(ds_{n+1})\ldots F(ds_1)$$

$$=\begin{cases}\displaystyle\int_t^\infty F(ds_1)p(s_1) & \text{if } n=1,\\ \displaystyle\int_t^\infty p(s_n)\int_0^{s_n} q(s_{n-1})\bar{F}^{-1}(s_{n-1})\int_0^{s_{n-1}}\ldots\int_0^{s_2} q(s_1)\bar{F}^{-1}(s_1)F(ds_1)\ldots F(ds_n) & \text{if } n\geq 2,\end{cases}$$

which follows by integrating $F(ds_{n+1})$ and then reversing the order of integration.

Hence, for $n\geq 1$ and $t\geq 0$, we have

$$P\{Y>t,\ \nu=n+1\} = \int_t^\infty p(x)\ \frac{1}{(n-1)!}\ [\int_0^x q(y)\bar{F}^{-1}(y)F(dy)]^{n-1}F(dx),$$

so that

$$P\{Y>t;\ \nu<+\infty\} = \sum_{n=1}^\infty P\{Y>t,\ \nu=n+1\}$$
$$= \int_t^\infty p(x)\ \exp\{\int_0^x q(y)\bar{F}^{-1}(y)F(dy)\}F(dx).$$

But since

$$\exp\{\int_0^x q(y)\bar{F}^{-1}(y)F(dy)\} = \exp\{\int_0^x [1-p(y)]\bar{F}^{-1}(y)F(dy)\}$$
$$= \bar{F}^{-1}(x)\ \exp\{-\int_0^x p(y)\bar{F}^{-1}(y)F(dy)\},$$

we have

$$P\{Y > t;\ \nu < +\infty\} = \int_t^\infty p(x)\ \exp\{-\int_0^x p(y)\bar{F}^{-1}(y)F(dy)\}\bar{F}^{-1}(x)F(dx)$$

$$= \exp\{-\int_0^t p(y)\bar{F}^{-1}(y)F(dy)\} - \exp\{-\int_0^\infty p(y)\bar{F}^{-1}(y)F(dy)\}.$$

Consequently for t=0,

$$P\{\nu < +\infty\} = 1 - \exp\{-\int_0^\infty p(y)\bar{F}^{-1}(y)F(dy)\}.$$

Now (c) and (d) follow immediately from the last two equalities.

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(A.6) Remark. In the above calculations we are making use of the fact that if H is a continuous distribution function on $(-\infty, +\infty)$ and ϕ is any function which is absolutely continuous with respect to Lebesgue measure on the range of H, then for all a,b

$$\int_a^b \phi'(H(x))H(dx) = \phi(H(b)) - \phi(H(a)).$$

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A stochastic model is developed to described the operation in time of the following maintained system setting. A piece of equipment is put in operation at time 0. Each time it fails, a maintenance action is taken which, with probability $p(t)$, is a complete repair or, with probability $q(t) = 1 - p(t)$, is a minimal repair, where t is the age of the equipment in use at the failure time. It is assumed that complete repair stores the equipment to its good as new condition, that minimal repair restores the equipment to its condition just prior to failure and that both maintenance actions take negligible time.

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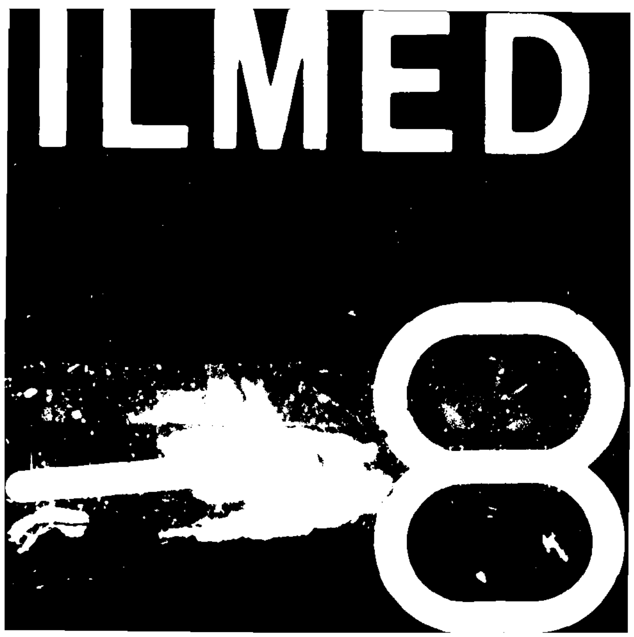
20.

If the equipment's life distribution F is a continuous function, the successive complete repair times are shown to be a renewal process with interarrival distribution $F_p(t) = 1 - \exp\{\int_0^t p(x)\bar{F}^{-1}(x)\, F(dx)\}$ for $t \geq 0$. Preservation and monotone properties of the model extending the results of Brown and Proschan (1980) are obtained.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